

A NOTE ON THE VALUES OF THE WEIGHTED q -BERNSTEIN POLYNOMIALS AND MODIFIED q -GENOCCHI NUMBERS WITH WEIGHT α AND β VIA THE p -ADIC q -INTEGRAL ON \mathbb{Z}_p

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ABSTRACT. The rapid development of q -calculus has led to the discovery of new generalizations of Bernstein polynomials and Genocchi polynomials involving q -integers. The present paper deals with weighted q -Bernstein polynomials and q -Genocchi numbers with weight α and β . We apply the method of generating function and p -adic q -integral representation on \mathbb{Z}_p , which are exploited to derive further classes of Bernstein polynomials and q -Genocchi numbers and polynomials. To be more precise we summarize our results as follows, we obtain some combinatorial relations between q -Genocchi numbers and polynomials with weight α and β . Furthermore we derive an integral representation of weighted q -Bernstein polynomials of degree n on \mathbb{Z}_p . Also we deduce a fermionic p -adic q -integral representation of product weighted q -Bernstein polynomials of different degrees n_1, n_2, \dots on \mathbb{Z}_p and show that it can be written with q -Genocchi numbers with weight α and β which yields a deeper insight into the effectiveness of this type of generalizations. Our new generating function possess a number of interesting properties which we state in this paper

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

The q -calculus theory is a novel theory that is based on finite difference re-scaling. First results in q -calculus belong to Euler, who discovered Euler's Identities for q -exponential functions and Gauss, who discovered q -binomial formula. The systematic development of q -calculus begins from F. H. Jackson who 1908 reintroduced the Euler Jackson q -difference operator (Jackson, 1908). One of important branches of q -calculus is q -type special orthogonal polynomials. Also p -adic numbers were invented by Kurt Hensel around the end of the nineteenth century and these two branches of number theory jointed with the link of p -adic q -integral and developed. In spite of their being already one hundred years old, these special numbers and polynomials, for instance q -Bernstein numbers and polynomials, q -Genocchi numbers and polynomials and etc. are still today enveloped in an aura of mystery within the scientific community. The p -adic integral was used in mathematical physics, for instance, the functional equation of the q -zeta function, q -stirling numbers and q -Mahler theory of integration with respect to the ring \mathbb{Z}_p together with Iwasawa's p -adic q - L functions. Professor T. Kim [29], also studied on p -adic interpolation

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functions of special orthogonal polynomials. In during the last ten years, the q -Bernstein polynomials and q -Genocchi polynomials have attracted a lot of interest and have been studied from different angles along with some generalizations and modifications by a number of researchers. By using the p -adic invariant q -integral on \mathbb{Z}_p , Professor T. Kim in [26], constructed p -adic Bernoulli numbers and polynomials with weight α . After Seo and first author in [9], extended Kim's method for q -Genocchi numbers and polynomials and also they defined q -Genocchi numbers and polynomials with weight α and β . Our aim of this paper is to show that a fermionic p -adic q -integral representation of product weighted q -Bernstein polynomials of different degrees n_1, n_2, \dots on \mathbb{Z}_p can be written with q -Genocchi numbers with weight α and β .

Let p be a fixed odd prime number. Throughout this paper we use the following notations. By \mathbb{Z}_p we denote the ring of p -adic rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{Q}_p denotes the field of p -adic rational numbers, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let \mathbb{N} be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The p -adic absolute value is defined by $|p|_p = \frac{1}{p}$. In this paper we assume $|q - 1|_p < 1$ as an indeterminate. In [23-25], let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the fermionic p -adic q -integral on \mathbb{Z}_p is defined by T. Kim:

$$(1.1) \quad \begin{aligned} I_{-q}(f) &= \int_{\mathbb{Z}_p} f(\xi) d\mu_{-q}(\xi) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{\xi=0}^{p^N-1} q^\xi f(\xi) (-1)^\xi. \end{aligned}$$

For $\alpha, k, n \in \mathbb{N}^*$ and $x \in [0, 1]$, T. Kim et al. defined weighted q -Bernstein polynomials as follows:

$$(1.2) \quad B_{k,n}^{(\alpha)}(x, q) = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}, \quad (\text{for detail, see [3, 27, 33, 34]}).$$

In (1.2), we put $q \rightarrow 1$ and $\alpha = 1$, $[x]_{q^\alpha}^k \rightarrow x^k$, $[1-x]_{q^{-\alpha}}^{n-k} \rightarrow (1-x)^{n-k}$ and we obtain the classical Bernstein polynomials (see[1], [2]),

where, $[x]_q$ is a q -extension of x which is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (\text{see [2-28, 32-34]}).$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$.

In previous paper [8], for $n \in \mathbb{N}^*$, modified q -Genocchi numbers with weight α and β are defined by Araci et al. as follows:

$$(1.3) \quad \begin{aligned} \frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} &= \int_{\mathbb{Z}_p} q^{-\beta\xi} [x + \xi]_{q^\alpha}^n d\mu_{-q^\beta}(\xi) \\ &= \frac{[2]_{q^\beta}}{[\alpha]_q^n (1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha\ell x} \frac{1}{1 + q^{\alpha\ell}} \\ &= [2]_{q^\beta} \sum_{m=0}^{\infty} (-1)^m [m+x]_{q^\alpha}^n. \end{aligned}$$

In the special case, $x = 0$, $g_{n,q}^{(\alpha,\beta)}(0) = g_{n,q}^{(\alpha,\beta)}$ are called the q -Genocchi numbers with weight α and β .

In [8], for $\alpha \in \mathbb{N}^*$ and $n \in \mathbb{N}$, q -Genocchi numbers with weight α and β are defined by Araci et al. as follows:

$$(1.4) \quad g_{0,q}^{(\alpha,\beta)} = 0, \text{ and } g_{n,q}^{(\alpha,\beta)}(1) + g_{n,q}^{(\alpha,\beta)} = \begin{cases} [2]_{q^\beta}, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

In this paper, we obtained some relations between the weighted q -Bernstein polynomials and the modified q -Genocchi numbers with weight α and β . From these relations, we derive some interesting identities on the q -Genocchi numbers with weight α and β .

2. ON THE q -GENOCCHI NUMBERS AND POLYNOMIALS WITH WEIGHT α AND β

By the definition of q -Genocchi polynomials with weight α and β , we easily get

$$\begin{aligned} \frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1} &= \int_{\mathbb{Z}_p} q^{-\beta\xi} [x + \xi]_{q^\alpha}^n d\mu_{-q^\beta}(\xi) \\ &= \int_{\mathbb{Z}_p} q^{-\beta\xi} \left([x]_{q^\alpha} + q^{\alpha x} [\xi]_{q^\alpha} \right)^n d\mu_{-q}(\xi) \\ &= \sum_{k=0}^n \binom{n}{k} [x]_{q^\alpha}^{n-k} q^{\alpha k x} \int_{\mathbb{Z}_p} q^{-\beta\xi} [\xi]_{q^\alpha}^k d\mu_{-q}(\xi) \\ &= \sum_{k=0}^n \binom{n}{k} [x]_{q^\alpha}^{n-k} q^{\alpha k x} \frac{g_{k+1,q}^{(\alpha,\beta)}}{k+1}. \end{aligned}$$

Therefore, we obtain the following Theorem:

Theorem 1. For $n, \alpha, \beta \in \mathbb{N}^*$, we have

$$(2.1) \quad g_{n,q}^{(\alpha,\beta)}(x) = q^{-\alpha x} \sum_{k=0}^n \binom{n}{k} q^{\alpha k x} g_{k,q}^{(\alpha,\beta)} [x]_{q^\alpha}^{n-k},$$

Moreover,

$$(2.2) \quad g_{n,q}^{(\alpha,\beta)}(x) = q^{-\alpha x} \left(q^{\alpha x} g_q^{(\alpha,\beta)} + [x]_{q^\alpha} \right)^n,$$

by using the umbral (symbolic) convention $\left(g_q^{(\alpha,\beta)} \right)^n = g_{n,q}^{(\alpha,\beta)}$.

By expression of (1.3), we get

$$\begin{aligned} \frac{g_{n+1,q^{-1}}^{(\alpha,\beta)}(1-x)}{n+1} &= \int_{\mathbb{Z}_p} q^{\beta\xi} [1-x + \xi]_{q^{-\alpha}}^n d\mu_{-q^{-\beta}}(\xi) \\ &= \frac{[2]_{q^{-\beta}}}{(1-q^{-\alpha})^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{-\alpha l(1-x)} \frac{1}{1+q^{-\alpha l}} \\ &= (-1)^n q^{\alpha n - \beta} \left(\frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} \frac{1}{1+q^{\alpha l}} \right) \\ &= (-1)^n q^{\alpha n - \beta} \frac{g_{n+1,q}^{(\alpha,\beta)}(x)}{n+1}. \end{aligned}$$

Consequently, we obtain the following Theorem:

Theorem 2. *The following*

$$(2.3) \quad g_{n+1,q^{-1}}^{(\alpha,\beta)} (1-x) = (-1)^n q^{\alpha n - \beta} g_{n+1,q}^{(\alpha,\beta)} (x)$$

is true.

From expression of (2.2) and Theorem 1, we get the following Theorem:

Theorem 3. *The following identity holds*

$$g_{0,q}^{(\alpha,\beta)} = 0, \text{ and } q^{-\alpha} \left(q^\alpha g_q^{(\alpha,\beta)} + 1 \right)^n + g_{n,q}^{(\alpha,\beta)} = \begin{cases} [2]_{q^\beta}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing $\left(g_q^{(\alpha,\beta)} \right)^n$ by $g_{n,q}^{(\alpha,\beta)}$.

For $n, \alpha \in \mathbb{N}$, by Theorem 3, we note that

$$\begin{aligned} q^{2\alpha} g_{n,q}^{(\alpha,\beta)} (2) &= \left(q^\alpha \left(q^\alpha g_q^{(\alpha,\beta)} + 1 \right) + 1 \right)^n \\ &= \sum_{k=0}^n \binom{n}{k} q^{k\alpha} \left(q^\alpha g_q^{(\alpha,\beta)} + 1 \right)^k \\ &= \left(q^\alpha g_q^{(\alpha,\beta)} + 1 \right)^0 + n q^\alpha \left(q^\alpha g_q^{(\alpha,\beta)} + 1 \right)^1 \\ &\quad + \sum_{k=2}^n \binom{n}{k} q^{k\alpha} \left(q^\alpha g_q^{(\alpha,\beta)} + 1 \right)^k \\ &= n q^{2\alpha} [2]_{q^\beta} - q^\alpha \sum_{k=0}^n \binom{n}{k} q^{\alpha k} g_{k,q}^{(\alpha,\beta)} \\ &= n q^{2\alpha} [2]_{q^\beta} + q^\alpha g_{n,q}^{(\alpha,\beta)}, \text{ if } n > 1. \end{aligned}$$

Consequently, we state the following Theorem:

Theorem 4. *For $n \in \mathbb{N}$, we have*

$$g_{n,q}^{(\alpha,\beta)} (2) = n [2]_{q^\beta} + \frac{g_{n,q}^{(\alpha,\beta)}}{q^\alpha}.$$

From expression of Theorem 2 and (2.3), we easily see that

$$\begin{aligned} (2.4) \quad & (n+1) q^{-\beta} \int_{\mathbb{Z}_p} q^{-\beta \xi} [1 - \xi]_{q^{-\alpha}}^n d\mu_{-q^\beta} (\xi) \\ &= (-1)^n q^{n\alpha - \beta} \int_{\mathbb{Z}_p} q^{-\beta \xi} [\xi - 1]_{q^\alpha}^n d\mu_{-q^\beta} (\xi) \\ &= (-1)^n q^{n\alpha - \beta} g_{n+1,q}^{(\alpha,\beta)} (-1) = g_{n+1,q^{-1}}^{(\alpha,\beta)} (2). \end{aligned}$$

Thus, we obtain the following Theorem.

Theorem 5. *The following identity*

$$(n+1) q^{-\beta} \int_{\mathbb{Z}_p} q^{-\beta \xi} [1 - \xi]_{q^{-\alpha}}^n d\mu_{-q^\beta} (\xi) = g_{n+1,q^{-1}}^{(\alpha,\beta)} (2)$$

is true.

Let $n, \alpha \in \mathbb{N}$. By expression of Theorem 4 and Theorem 5, we get

$$(2.5) \quad \begin{aligned} & (n+1) q^{-\beta} \int_{\mathbb{Z}_p} q^{-\beta\xi} [1-\xi]_{q^{-\alpha}}^n d\mu_{-q^\beta}(\xi) \\ &= (n+1) q^{-\beta} [2]_{q^\beta} + q^\alpha g_{n+1, q^{-1}}^{(\alpha, \beta)}. \end{aligned}$$

For (2.5), we obtain corollary as follows:

Corollary 1. *For $n, \alpha \in \mathbb{N}^*$, we have*

$$\int_{\mathbb{Z}_p} q^{-\beta\xi} [1-\xi]_{q^{-\alpha}}^n d\mu_{-q^\beta}(\xi) = [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n+1, q^{-1}}^{(\alpha, \beta)}}{n+1}.$$

3. NOVEL IDENTITIES ON THE WEIGHTED q -GENOCCHI NUMBERS

In this section, we develop modified q -Genocchi numbers with weight α and β , namely, we derive interesting and worthwhile relations in Analytic Number Theory.

For $x \in \mathbb{Z}_p$, the p -adic analogues of weighted q -Bernstein polynomials are given by

$$(3.1) \quad B_{k,n}^{(\alpha)}(x, q) = \binom{n}{k} [x]_{q^\alpha}^k [1-x]_{q^{-\alpha}}^{n-k}, \text{ where } n, k, \alpha \in \mathbb{N}^*.$$

By expression of (3.1), Kim et. al. get the symmetry of q -Bernstein polynomials weight α as follows:

$$(3.2) \quad B_{k,n}^{(\alpha)}(x, q) = B_{n-k,n}^{(\alpha)}(1-x, q^{-1}), \text{ (for detail, see [27]).}$$

Thus, from Corollary 1, (3.1) and (3.2), we see that

$$\begin{aligned} \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(\xi, q) q^{-\beta\xi} d\mu_{-q^\beta}(\xi) &= \int_{\mathbb{Z}_p} B_{n-k,n}^{(\alpha)}(1-\xi, q^{-1}) q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} q^{-\beta\xi} [1-\xi]_{q^{-\alpha}}^{n-l} d\mu_{-q^\beta}(\xi) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n-l+1, q^{-1}}^{(\alpha, \beta)}}{n-l+1} \right). \end{aligned}$$

For $n, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n > k$, we obtain

$$(3.3) \quad \begin{aligned} & \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(\xi, q) q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ &= \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n-l+1, q^{-1}}^{(\alpha, \beta)}}{n-l+1} \right) \\ &= \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n+1, q^{-1}}^{(\alpha, \beta)}}{n+1}, & \text{if } k = 0, \\ \binom{n}{k} \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n-l+1, q^{-1}}^{(\alpha, \beta)}}{n-l+1} \right), & \text{if } k > 0. \end{cases} \end{aligned}$$

Let us take the fermionic p -adic q -integral on \mathbb{Z}_p on the weighted q -Bernstein polynomials of degree n as follows:

$$(3.4) \quad \int_{\mathbb{Z}_p} B_{k,n}^{(\alpha)}(\xi, q) q^{-\beta\xi} d\mu_{-q^\beta}(\xi) = \binom{n}{k} \int_{\mathbb{Z}_p} q^{-\beta\xi} [\xi]_{q^\alpha}^k [1-\xi]_{q^{-\alpha}}^{n-k} d\mu_{-q^\beta}(\xi) \\ = \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{g_{l+k+1,q}^{(\alpha,\beta)}}{l+k+1}.$$

Consequently, by expression of (3.3) and (3.4), we state the following Theorem:

Theorem 6. *The following identity holds*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \frac{g_{l+k+1,q}^{(\alpha,\beta)}}{l+k+1} = \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n+1,q-1}^{(\alpha,\beta)}}{n+1}, & \text{if } k=0, \\ \sum_{l=0}^k \binom{k}{l} (-1)^{k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n-l+1,q-1}^{(\alpha,\beta)}}{n-l+1} \right), & \text{if } k>0. \end{cases}$$

Let $n_1, n_2, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n_1 + n_2 > 2k$. Then, we get

$$\int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(\xi, q) B_{k,n_2}^{(\alpha)}(\xi, q) q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ = \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \int_{\mathbb{Z}_p} q^{-\beta\xi} [1-\xi]_{q^{-\alpha}}^{n_1+n_2-l} d\mu_{-q^\beta}(\xi) \\ = \left(\binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2-l+1,q-1}^{(\alpha,\beta)}}{n_1+n_2-l+1} \right) \right) \\ = \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+1}, & \text{if } k=0, \\ \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2-l+1,q-1}^{(\alpha,\beta)}}{n_1+n_2-l+1} \right), & \text{if } k \neq 0. \end{cases}$$

Therefore, we obtain the following Theorem:

Theorem 7. *For $n_1, n_2, k \in \mathbb{N}^*$ and $\alpha, \beta \in \mathbb{N}$ with $n_1 + n_2 > 2k$, we have*

$$\int_{\mathbb{Z}_p} q^{-\beta\xi} B_{k,n_1}^{(\alpha)}(\xi, q) B_{k,n_2}^{(\alpha)}(\xi, q) d\mu_{-q^\beta}(\xi) \\ = \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+1}, & \text{if } k=0, \\ \binom{n_1}{k} \binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2-l+1,q-1}^{(\alpha,\beta)}}{n_1+n_2-l+1} \right), & \text{if } k \neq 0. \end{cases}$$

By using the binomial theorem, we can derive the following equation.

$$(3.5) \quad \int_{\mathbb{Z}_p} B_{k,n_1}^{(\alpha)}(\xi, q) B_{k,n_2}^{(\alpha)}(\xi, q) q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ = \prod_{i=1}^2 \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \int_{\mathbb{Z}_p} [\xi]_{q^\alpha}^{2k+l} q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ = \prod_{i=1}^2 \binom{n_i}{k} \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{g_{l+2k+1,q}^{(\alpha,\beta)}}{l+2k+1}.$$

Thus, we can obtain the following Corollary:

Corollary 2. *For $n_1, n_2, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $n_1 + n_2 > 2k$, we have*

$$\begin{aligned} & \sum_{l=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{l} (-1)^l \frac{g_{l+2k+1,q}^{(\alpha,\beta)}}{l+2k+1} \\ &= \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+1}, & \text{if } k=0, \\ \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{2k+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2-l+1,q-1}^{(\alpha,\beta)}}{n_1+n_2-l+1} \right), & \text{if } k \neq 0. \end{cases} \end{aligned}$$

For $\xi \in \mathbb{Z}_p$ and $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $\sum_{l=1}^s n_l > sk$. Then we take the fermionic p -adic q -integral on \mathbb{Z}_p for the weighted q -Bernstein polynomials of degree n as follows:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \underbrace{B_{k,n_1}^{(\alpha)}(\xi, q) B_{k,n_2}^{(\alpha)}(\xi, q) \dots B_{k,n_s}^{(\alpha)}(\xi, q)}_{s\text{-times}} q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ &= \prod_{i=1}^s \binom{n_i}{k} \int_{\mathbb{Z}_p} [\xi]_{q^\alpha}^{sk} [1-\xi]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-sk} q^{-\beta\xi} d\mu_{-q^\beta}(\xi) \\ &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{l+sk} \int_{\mathbb{Z}_p} q^{-\beta\xi} [1-\xi]_{q^{-\alpha}}^{n_1+n_2+\dots+n_s-sk} d\mu_{-q^\beta}(\xi) \\ &= \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+\dots+n_s+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+\dots+n_s+1}, & \text{if } k=0, \\ \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+\dots+n_s-l+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+\dots+n_s-l+1} \right), & \text{if } k \neq 0. \end{cases} \end{aligned}$$

So from above, we have the following Theorem:

Theorem 8. *For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $\sum_{l=1}^s n_l > sk$. Then we have*

$$\begin{aligned} & \int_{\mathbb{Z}_p} q^{-\beta\xi} \prod_{i=1}^s B_{k,n_i}^{(\alpha)}(\xi) d\mu_{-q}(\xi) \\ &= \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+\dots+n_s+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+\dots+n_s+1}, & \text{if } k=0, \\ \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+\dots+n_s-l+1,q-1}^{(\alpha,\beta)}}{n_1+n_2+\dots+n_s-l+1} \right), & \text{if } k \neq 0. \end{cases} \end{aligned}$$

From the definition of weighted q -Bernstein polynomials and the binomial theorem, we easily get

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} q^{-\beta\xi} \underbrace{B_{k,n_1}^{(\alpha)}(\xi, q) B_{k,n_2}^{(\alpha)}(\xi, q) \dots B_{k,n_s}^{(\alpha)}(\xi, q)}_{s\text{-times}} d\mu_{-q^\beta}(\xi) \\
 &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^s (n_d - k)}{l} (-1)^l \int_{\mathbb{Z}_p} q^{-\beta\xi} [\xi]_{q^\alpha}^{sk+l} d\mu_{-q^\beta}(\xi) \\
 (3.6) &= \prod_{i=1}^s \binom{n_i}{k} \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^s (n_d - k)}{l} (-1)^l \frac{g_{l+sk+1,q}^{(\alpha,\beta)}}{l+sk+1}.
 \end{aligned}$$

Therefore, from (3.6) and Theorem 8, we get interesting Corollary as follows:

Corollary 3. For $s \in \mathbb{N}$ with $s \geq 2$, let $n_1, n_2, \dots, n_s, k \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}$ with $\sum_{l=1}^s n_l > sk$. We have

$$\begin{aligned}
 & \sum_{l=0}^{n_1+\dots+n_s-sk} \binom{\sum_{d=1}^s (n_d - k)}{l} (-1)^l \frac{g_{l+sk+1,q}^{(\alpha,\beta)}}{l+sk+1} \\
 &= \begin{cases} [2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+\dots+n_s+1,q^{-1}}^{(\alpha,\beta)}}{n_1+n_2+\dots+n_s+1}, & \text{if } k = 0, \\ \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk+l} \left([2]_{q^\beta} + q^{\alpha-\beta} \frac{g_{n_1+n_2+\dots+n_s-l+1,q^{-1}}^{(\alpha,\beta)}}{n_1+n_2+\dots+n_s-l+1} \right), & \text{if } k \neq 0. \end{cases}
 \end{aligned}$$

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